JOURNAL OF APPROXIMATION THEORY 1, 180–181 (1968)

On N-Parameter and Unisolvent Families

RICHARD BARRAR AND HENRY LOEB

Mathematics Department, University of Oregon, Eugene, Oregon 97403

Dunham [1] in a recent note has pointed out a gap in the proofs of Tornheim [2] and Rice [3] in their important papers on Chebyshev approximation. The gap occurs since neither paper considers the possibility that the error curve of the best approximation is a nonzero constant.

In this note we will prove by an elementary argument that this possibility cannot occur in the case considered by Tornheim or in the unisolvent case considered by Rice when the degree of unisolvence is 1, 2 or 3.

Thus in Rice's notation, let $F(a^*,x)$ be the best approximation to $g(x)$ on $[0,1]$ with $g(x) - F(a^*,x) \equiv c \neq 0$ where clearly we may assume the constant c is positive, and let the degree of unisolvence be 1. Then for a given $\epsilon > 0$ there is an $F(a,x)$ such that:

(i) $\max_{x\in[0,1]} |F(a^*,x) - F(a,x)| < \epsilon$

(ii) $F(a,x)$ doesn't intersect $F(a^*,x)$

(iii) $F(a,0) - F(a^*,0) = \delta > 0.$

If ϵ and δ are less than $c/2$, $F(a,x)$ is a better approximation to $g(x)$ than $F(a^*,x)$, which is a contradiction. This proof can be modified slightly to handle the case when the degree of unisolvence is 2 or 3. For example, in the latter situation, set $F(a,0) = F(a^*,0)$, $F(a,1) = F(a^*,1)$, $F(a,\frac{1}{2}) - F(a^*,\frac{1}{2}) = \delta$. Then $F(a,x)$ is also a best approximation where the corresponding error curve is nonconstant and does not alternate. Theorem 2 of Rice's work [1] is applicable to this situation, and shows that $F(a,x)$ cannot be a best approximation, which is again a contradiction.

Since Tornheim considered a special case of unisolvency, the above proof is applicable. However, in the case considered by Tornheim we are able to apply induction to remove the possibility of a best approximation yielding a non-zero constant error curve.

Let F be the N-Parameter family under consideration. Thus, assume the result holds for any $(N-1)$-parameter family. We will show that if an $f \in F$ is a best approximation to g from F with the property $g - f \equiv c > 0$ then we are led to a contradiction. Let f, g, c have the above stated properties and set $\hat{F} = \{f' \in F: f'(1) = f(1)\}$. For $0 < \epsilon < 1$, $\hat{F}$ is an $(N-1)$-parameter family over $[0, 1-\epsilon]$. Let f_ϵ be the best approximation to g from $\hat{F}$ over $[0, 1-\epsilon]$. By induction and Tornheim's results, $g - f_\epsilon$ has a zero in $[0, 1-\epsilon]$. Note that if

$$\|\phi\|_\epsilon = \max_{x \in [0,\, 1-\epsilon]} |\phi(x)|,$$

then for $\epsilon \leqslant \epsilon_0$, $\|g - f_\epsilon\|_{\epsilon 0} \leqslant \|g - f_\epsilon\|_\epsilon < c$. Hence by a compactness argument using Tornheim's Theorem 5, one can assume

$$\lim_{\epsilon \to 0} \|f_\epsilon - \hat{f}\|_0 = 0$$

where $\hat{f} \in \hat{F}$. The claim is made that $\hat{f}$ is also a best approximation to g from F over $[0, 1]$. If the claim is false, there is an $x \in [0, 1)$ such that $|g(x) - \hat{f}(x)| > c$. But for small ϵ, $x \in [0, 1-\epsilon]$ which implies $|g(x) - f_\epsilon(x)| < c$. Taking the limit, a contradiction is reached. It is easy to see that $g - \hat{f}$ is a nonconstant error curve. By Tornheim's results $g - \hat{f}$ must alternate N times. Therefore, by a standard uniqueness argument, $f \equiv \hat{f}$, which is a contradiction.

References

1. C. Dunham, Necessity of alternation. To appear in *Can. Math. Bull.*
2. L. Tornheim, On N-parameter families of functions and associated convex functions. *Trans. Am. Math. Soc.* **69** (1950), 457–467.
3. J. R. Rice, Tchebycheff approximation by functions unisolvent of variable degree. *Trans. Math. Soc.* **99** (1961), 298–302.